

# Centrifugal deformations of the gravitational kink

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## Abstract

The Kaluza-Klein reduction of 4d conformally flat spacetimes is reconsidered. The corresponding 3d equations are shown to be equivalent to 2d gravitational kink equations augmented by a centrifugal term. For space-like gauge fields and non-trivial values of the centrifugal term the gravitational kink solutions describe a spacetime that is divided in two disconnected regions.

*Key words:* Gravitational kinks, conformal flatness, Kaluza-Klein reduction.

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## 1 Introduction

Reducing the gravitational Chern-Simons term from three to two dimensions produces an interesting as well as surprising outcome [1]. On the one hand the resulting equations of motion require the vanishing of the 3d Cotton tensor and, hence, impose conformal flatness of the original 3d spacetime. On the other hand, they take the form of general covariant kink equations that, besides symmetry breaking solutions, support an associated kink. Let us parameterize the 3d line element as  $ds_{(3)}^2 = g_{ij}dx^i dx^j + (A_i dx^i + dx^2)^2$ , where Latin indices range over 0, 1 and all quantities are independent of the third coordinate  $x^2$ . Then, the equations obtained by Guralnik, Iorio, Jackiw and Pi ((4.47,48,49) in Ref. [1]) read

$$R = k + 3f^2, \tag{1a}$$

$$D^2 f + kf - f^3 = 0, \tag{1b}$$

$$D_i D_j f - \frac{1}{2} g_{ij} D^2 f = 0. \tag{1c}$$

Here  $g_{ij}$  is the 2d spacetime metric,  $D_i$  is the associated covariant derivative,  $D^2 = D_i D^i$ ,  $R$  is the relative scalar curvature and  $k$  is an arbitrary constant. The Kaluza-Klein field strength  $F_{ij} = \partial_i A_j - \partial_j A_i$  has been expressed in terms of its dual scalar<sup>1</sup>

$$F_{ij} \equiv \sqrt{-g} \varepsilon_{ij} f \quad (2)$$

Equation (1b) is the covariant generalization of the flat space kink equation with biquadratic potential

$$V(f) = \frac{1}{4}(k - f^2)^2, \quad (3)$$

supporting, for positive values of  $k$ , the well known kink profile. Equations (1a) and (1c) appear instead, as subsidiary constraints necessary to uniquely determine the geometry  $g_{ij}$  together with the field  $f$ . As shown in Ref. [1], general covariant kinks bear a close relation to flat space kinks governed by identical potentials. Indeed, equations (1) support the solution

$$g_{ij} dx^i dx^j = -\frac{1}{4} k^2 \operatorname{sech}^4\left(\frac{\sqrt{k}}{2} x^1\right) (dx^0)^2 + (dx^1)^2, \quad (4a)$$

$$A_i dx^i = \pm \frac{1}{2} k \operatorname{sech}^2\left(\frac{\sqrt{k}}{2} x^1\right) dx^0, \quad (4b)$$

with the corresponding kink profile

$$f = \pm \sqrt{k} \tanh\left(\frac{\sqrt{k}}{2} x^1\right). \quad (4c)$$

It is rather natural to wonder whether this situation generalizes to higher dimensions. Indeed, the Kaluza-Klein reduction of conformally flat spaces was recently considered by Grumiller and Jackiw for arbitrary dimensions [2]. The resulting, rather daunting, equations dramatically simplify for the reduction from 4d to 3d. Grumiller and Jackiw restricted to that case and constructed special solutions based on a further Ansatz on the 3d metric. Here, we reconsider these equations in their generality. For space-like gauge fields, we show that the 3d general covariant equations describing the Kaluza-Klein reduction of 4d conformally flat spaces correspond to the 2d general covariant kink equations (1) supplemented by a centrifugal term. Hence, besides the 3d extension of the gravitational kink (4), 4d conformally flat Kaluza-Klein spaces also support their ‘centrifugal deformations’.

Our discussion proceeds as follows. In §2 we briefly review 3d Grumiller-Jackiw equations and supplement them with some extra considerations. In §3 we

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<sup>1</sup> Some signs differ from Ref. [1], and in what follow from Ref. [2], due to different conventions. Here  $R_{ijk}{}^l = \partial_i \Gamma_{jk}^l - \dots$ ,  $R_{ij} = R_{kij}{}^k$ ,  $R = R_i{}^i$  and analogously in other dimensions. The single time-like direction carries negative signature, while space-like directions carry positive signature.

take advantage of general covariance to adapt coordinates and reduce the equations to 2d centrifugal kink equations. The centrifugal deformations of the gravitational kink are eventually constructed in §4.

## 2 Grumiller-Jackiw equations in 3d

Let us parameterize the 4d line element as  $ds_{(4)}^2 = g_{\mu\nu}dx^\mu dx^\nu + (A_\mu dx^\mu + dx^3)^2$ , where Greek indices range over 0, 1, 2 and all quantities are independent of the last coordinate  $x^3$ . Then, the 3d Grumiller-Jackiw equations<sup>2</sup> ((22a,b) in Ref. [2]) read

$$R_{\mu\nu} - \frac{1}{3}Rg_{\mu\nu} = f_\mu f_\nu - \frac{1}{3}f_\kappa f^\kappa g_{\mu\nu}, \quad (5a)$$

$$D_\mu f_\nu + D_\nu f_\mu = 0. \quad (5b)$$

Here,  $g_{\mu\nu}$  is the 3d spacetime metric,  $D_\mu$  is the associated covariant derivative and  $R_{\mu\nu}$  and  $R$  are the corresponding Ricci and scalar curvatures. The Kaluza-Klein field strength  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  has been reexpressed in terms of its dual vector

$$F_{\mu\nu} \equiv \sqrt{-g}\varepsilon_{\mu\nu\kappa} f^\kappa. \quad (6)$$

Grumiller and Jackiw also derived two important consequences of equations (5a) and (5b). First, the scalar curvature  $R$  can be expressed in terms of  $f_\mu f^\mu$  and an arbitrary constant  $k$  as ((23) in Ref. [2])

$$R = 3k - 5f_\mu f^\mu. \quad (7)$$

Second, the vector  $F^\mu = \frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu\kappa}D_\nu f_\kappa$ , when not identically vanishing, is a second Killing vector of the geometry ((26c) in Ref. [2])

$$D_\mu F_\nu + D_\nu F_\mu = 0. \quad (8)$$

Before proceeding it is useful to rewrite equation (8) in a slightly different form and to supplement equations (5a) and (5b) with a further integrability condition.

After rising both indices of (8) we reexpress  $F^\mu$  in terms of  $f^\mu$  obtaining

$$\varepsilon^{\lambda\rho\sigma}D^\kappa D_\rho f_\sigma + \varepsilon^{\kappa\rho\sigma}D^\lambda D_\rho f_\sigma = 0.$$

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<sup>2</sup> In 3d, Grumiller-Jackiw equations are the Minkowskian analogue of the Euclidean Einstein-Weyl equations, widely studied and completely solved in mathematics (see [3], [4] and references therein).

Contracting with  $\varepsilon_{\lambda\mu\nu}$ , reexpressing  $\varepsilon$ -symbols products in terms of Kronecker deltas and taking into account (5b) we immediately have

$$D_\kappa D_\mu f_\nu - \frac{1}{2}g_{\kappa\mu} D^2 f_\nu + \frac{1}{2}g_{\kappa\nu} D^2 f_\mu = 0, \quad (9)$$

closely resembling the traceless equation (1c).

To derive the integrability condition consider the covariant derivative of (5b)

$$D_\kappa D_\mu f_\nu + D_\kappa D_\nu f_\mu = 0.$$

Antisymmetrizing in  $\kappa$  and  $\mu$ , contracting  $\mu$  with  $\nu$  and taking into account the vanishing of the covariant divergence of the gauge field,  $D_\mu f^\mu = 0$ , we have

$$[D_\kappa, D_\mu] f^\mu - D_\mu D^\mu f_\kappa = 0.$$

Expressing the commutator of covariant derivatives in terms of the Ricci tensor and inserting (5a) and (7), we eventually obtain the integrability condition

$$D^2 f^\mu + k f^\mu - f_\nu f^\nu f^\mu = 0, \quad (10)$$

closely resembling the gravitational kink equation (1b). The similarity between the sets of equations (7), (9), (10) and (1a), (1c), (1b) is made even stronger by properly adapted coordinates.

### 3 Darboux coordinates

By an appropriate coordinate transformation we now set  $f^0 = f^1 = 0$ , i.e.  $F_{12} = F_{20} = 0$ . The existence of such a coordinate frame is guarantied by a classical result in symplectic geometry, Darboux theorem [5]. In 3d it ensures the possibility of finding, in a *finite* neighborhood of every point, local coordinates in such a way that a given closed two-form (a  $U(1)$  gauge field) can be rewritten, e.g., as

$$F_{\mu\nu} = \begin{pmatrix} F_{ij} & 0 \\ 0 & 0 \end{pmatrix} \quad (11)$$

with  $i, j = 0, 1$  and  $F_{ij}$  a closed 2d two-form. Darboux coordinates are determined up to the reparametrization  $x^i \rightarrow x'^i(x^0, x^1)$ ,  $x^2 \rightarrow x'^2(x^0, x^1, x^2)$ , so that we still have a certain freedom in choosing them. In the adapted frames the metric takes an arbitrary form. Without loss of generality it can be parameterized as

$$g_{\mu\nu} dx^\mu dx^\nu = g_{ij} dx^i dx^j + h (a_i dx^i + dx^2)^2, \quad (12)$$

where  $g_{ij}$ ,  $a_i$  and  $h$ , are arbitrary functions of all coordinates.<sup>3</sup> Under the residual covariance group,  $g_{ij}$  transforms like a 2d metric tensor,  $a_i$  identifies with a 2d gauge potential taking values in the 1d diffeomorphism algebra, while  $h$  transforms like a scalar. For  $f^\mu$  space-like,  $g_{ij}$  is Minkowskian,  $h$  definite positive and time can be identified with  $x^0$ . For  $f^\mu$  time-like,  $g_{ij}$  is Euclidean,  $h$  definite negative and time has to be identified with  $x^2$ . In any case, we proceed as in §1 and re-express the effectively 2d gauge field  $F_{ij} = \partial_i A_j - \partial_j A_i$  in terms of its dual scalar<sup>4</sup>

$$F_{ij} \equiv \sqrt{|g|} \varepsilon_{ij} f. \quad (13)$$

Consequently, in adapted coordinates, the 3d gauge field  $f^\mu$  rewrites as

$$f^\mu = (0, 0, |h|^{-\frac{1}{2}} f). \quad (14)$$

We also introduce the gauge curvature  $f_{ij}$  associated to the, in general, non-abelian vector potential  $a_i$

$$f_{ij} = \partial_i a_j - \partial_j a_i - a_i \partial_2 a_j + a_j \partial_2 a_i \quad (15)$$

and the relative dual scalar

$$f_{ij} \equiv \sqrt{|g|} \varepsilon_{ij} f. \quad (16)$$

Eventually, we take advantage of residual covariance by rescaling  $x^2$  in such a way that

$$h = h(x^0, x^1). \quad (17)$$

Correspondingly, covariance is reduced to arbitrary redefinitions of  $x^i$ ,  $x^i \rightarrow x'^i(x^0, x^1)$ , and linear redefinitions of  $x^2$ ,  $x^2 \rightarrow \xi(x^0, x^1)x^2$ .

We now proceed by rewriting equations (5), (9) and (10) in adapted Darboux coordinates. The most convenient starting point is equation (5b). Its  $ij$ ,  $i2$  and  $22$  components can be rearranged into the lower dimensional covariant equations

$$\partial_2 g_{ij} = 0, \quad (18a)$$

$$\partial_2 a_i = \frac{1}{2} \partial_i \ln(|h| f^{-2}), \quad (18b)$$

$$\partial_2 f = 0. \quad (18c)$$

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<sup>3</sup> We remark that, in spite of the similarity, this is not a further Kaluza-Klein Ansatz as the metric entries depend on the ‘extra’ coordinate  $x^2$ .

<sup>4</sup> With the only exceptions of coordinates and of the gauge field  $f$ , 2d quantities are expressed in Roman characters.

The first and third clearly require

$$g_{ij} = g_{ij}(x^0, x^1), \quad f = f(x^0, x^1), \quad (19)$$

while the integration of the second one produces

$$a_i = \frac{1}{2} \partial_i \ln(|h|f^{-2}) x^2 + \bar{a}_i(x^0, x^1)$$

with  $\bar{a}_i(x^0, x^1)$  an arbitrary 2d one-form. The gauge potential,  $a_i$ , is at most linear in the ‘extra’ coordinate  $x^2$ . As a consequence, we can take further advantage of residual covariance to eliminate the  $x^2$  dependence. In fact, the coordinate transformation  $x^2 \rightarrow \frac{1}{2} \ln(|h|f^{-2}) x^2$  causes the off-diagonal blocks of the metric to transform as  $a_i \rightarrow a_i - \frac{1}{2} \partial_i \ln(|h|f^{-2}) x^2$ , making the transformed gauge potential to be independent of  $x^2$ ,

$$a_i = a_i(x^0, x^1). \quad (20)$$

In the new adapted coordinates we have  $\partial_2 a_i = 0$  so equation (18b) implies the proportionality between  $h$  and  $f^2$ . The relative scale factor can be fixed to any non-zero value by a constant rescaling of  $x^2$ . It is positive/negative when  $f^\mu$  is space/time-like. For definiteness, we fix the scale factor to be equal to  $\pm 2$ , i.e.

$$h = \pm 2 f^2. \quad (21)$$

Here and in the following, upper/lower signs refer to space/time-like  $f^\mu$ . The ‘extra’ coordinate  $x^2$  is now completely fixed and residual covariance is reduced to 2d general covariance,  $x^i \rightarrow x'^i(x^0, x^1)$ .

Next, we consider the integrability condition (10). Taking (19), (20) and (21) into account, its  $i$  and 2 components become

$$\partial_i (f^3 f) = 0, \quad (22a)$$

$$D^2 f + k f \mp f^3 + f^3 f^2 = 0, \quad (22b)$$

with  $D^2 = D_i D^i$  and  $D_i$  the covariant derivative associated to the 2d metric  $g_{ij}$ . The first equation requires the product  $f^3 f$  to be constant, allowing to reexpress  $f$  in terms of  $f$ . We set

$$f^3 f = l \quad (23)$$

with  $l$  an arbitrary constant. Consequently, (22b) reduces to

$$D^2 f + k f \mp f^3 + \frac{l^2}{f^3} = 0. \quad (24a)$$

For space-like  $f^\mu$ , (24a) is the gravitational kink equation (1b) augmented by a centrifugal term of ‘angular momentum’  $l$ . Eventually, we consider equations

(5a) and (9) in view of (19), (20), (21) and (23). Equation (5a) reduces to

$$R = k \mp 3f^2 - \frac{3l^2}{f^4}, \quad (24b)$$

for  $R$  the 2d scalar curvature associated to  $g_{ij}$ . For space-like  $f^\mu$ , (24b) corresponds to the curvature constraint (1b) up to the ‘centrifugal’ term  $-3l^2/f^4$ . Equation (9) reduces instead to

$$D_i D_j f - \frac{1}{2} g_{ij} D^2 f = 0 \quad (24c)$$

corresponding to the traceless equation (1c) without any modification.

Equations (24) are completely equivalent to equations (5).

#### 4 Centrifugal kinks

We now specialize to space-like gauge fields. For  $l = 0$  equations (24) exactly correspond to the gravitational kink of equations (1). Therefore, they support a 3d gravitational kink, with  $g_{ij}$  given by (4a),  $A_i$  by (4b),  $a_i = 0$  and  $h$  equal to twice the square of (4c). The resulting 3d spacetime is a warped product of the 2d kink spacetime with the real line, where the warp factor is twice the square of the kink profile. For  $l \neq 0$ , the biquadratic potential (3) is supplemented by the centrifugal term  $l^2/(2f^2)$ . Correspondingly, the scalar curvature  $R$  is augmented by  $-3l^2/f^4 = -(l^2/(2f^2))''$  (see Appendix B of Ref. [1]). By properly choosing the integration constant the new potential can be written as

$$V(f) = \frac{(\kappa - \lambda - f^2)^2(f^2 + \lambda)}{4f^2}, \quad (25)$$

with the constants  $\kappa$  and  $\lambda$  related to  $k$  and  $l$  by  $k = \kappa - 3\lambda/2$  and  $2l^2 = (\kappa - \lambda)^2 \lambda$ . The integration of the corresponding flat space equation (see Appendices A and B of Ref. [1]) is immediate and leads to the following deformation of the general covariant kink

$$g_{ij} dx^i dx^j = - \frac{\kappa^3 \operatorname{sech}^4\left(\frac{\sqrt{\kappa}}{2} x^1\right) \tanh^2\left(\frac{\sqrt{\kappa}}{2} x^1\right)}{4\kappa \tanh^2\left(\frac{\sqrt{\kappa}}{2} x^1\right) - 4\lambda} (dx^0)^2 + (dx^1)^2, \quad (26a)$$

$$a_i dx^i = \pm \frac{\kappa \sqrt{2\lambda}}{4(\kappa - \lambda) \cosh^2\left(\frac{\sqrt{\kappa}}{2} x^1\right) - 4\kappa} dx^0, \quad (26b)$$

$$h = 2\kappa \tanh^2\left(\frac{\sqrt{\kappa}}{2} x^1\right) - 2\lambda, \quad (26c)$$

$$A_i dx^i = \pm \frac{\kappa}{2} \operatorname{sech}^2\left(\frac{\sqrt{\kappa}}{2} x^1\right) dx^0, \quad (26d)$$

with the corresponding centrifugal distortion of the kink profile

$$f = \pm \sqrt{\kappa \tanh^2 \left( \frac{\sqrt{\kappa}}{2} x^1 \right) - \lambda}. \quad (26e)$$

For  $l = 0$  the constants  $\kappa$  and  $\lambda$  respectively reduce to  $k$  and 0, so that (26) correctly reproduces the 3d gravitational kink. For  $l \neq 0$  the 2d metric  $g_{ij}$ , and the corresponding scalar curvature  $R$ , is singular at  $x^1 = \pm \frac{2}{\sqrt{\kappa}} \operatorname{arctanh} \sqrt{\frac{\lambda}{\kappa}}$ , while the gauge field  $f$  is only defined for  $|x^1| > \frac{2}{\sqrt{\kappa}} \operatorname{arctanh} \sqrt{\frac{\lambda}{\kappa}}$ . The effect of the centrifugal interaction is that of breaking the kink in two parts, pushing them apart, and thus dividing spacetime in two disconnected regions.

## Acknowledgments

We would like to thank Daniel Grumiller and Roman Jackiw for inspiring conversations. This work was supported by the Royal Society.

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